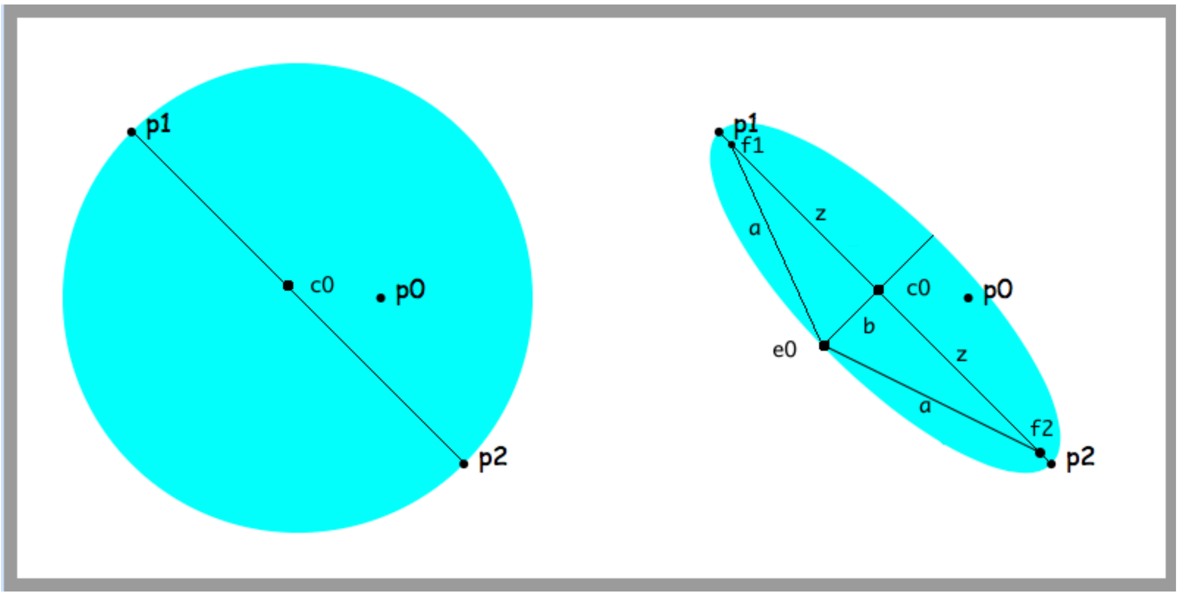
Mathematical background to Point.isBetween()



Point.isBetween1(p1, p2) tests whether this point (p0) lies on or within a circle centred midway between p1 and p2 with radius half the distance between p1 and p2. The circle passes through p1 and p2.

Consider points p1(x1, y1), p2(x2, y2) and p0(x0, y0).

Take centre point c0(cx, cy) with cx = (x1 + x2) / 2 and cy = (y1 + y2) / 2.

Take circle of radius r where r^2 = ((x1 - x2)^2 + (y1 - y2)^2) / 4.

p0 lies on or within the circle of radius r centred at c0 if

(x0 - cx)^2 + (y0 - cy)^2 <= r^2

giving the condition

(x0 - (x1 + x2)/2)^2 + (y0 - (y1 + y2) / 2)^2 <= ((x1 - x2)^2 + (y1 - y2)^2) / 4

which simplifies to

(x0 - x1)*(x0 - x2) + (y0 - y1)*(y0 - y2) <= 0

Point.isBetween2(p1, p2, e) tests whether this point (p0) lies on or within an ellipse of eccentricity e centred midway between p1 and p2 where points p1 and p2 are the end points of the major axis of the ellipse.

Consider points p1(x1, y1), p2(x2, y2) and p0(x0, y0).
Centre point c0(cx, cy)
Focus f1(f1x, f1y)
Focus f2(f2x, f2y)

Major semi-axis of length a
Minor semi-axis of length b

Distance c0 to f1 is z
Distance c0 to f2 is z

For the point e0 on the minor semi-axis and on the ellipse b^2 + z^2 = a^2

Let alfa = (a + z)/(2*a) and beta = 1 - alfa = (a - z)/(2*a).

alfa = (a + sqrt(a^2 - b^2))/(2*a) = (1 + sqrt(1 - b^2/a^2))/2

noting that e = sqrt(1 - b^2/a^2) gives alfa = (1 + e) / 2 and beta = (1 - e) / 2

then f1x = x1*alfa + x2*beta, f1y = y1*alfa + y2*beta
and f2x = x1*beta + x2*alfa, f2y = y1*beta + y2*alfa

d1 = distance p0 to f1 is sqrt((x0 - f1x)^2 + (y0 - f1y)^2)
d2 = distance p0 to f2 is sqrt((x0 - f2x)^2 + (y0 - f2y)^2)

The condition that p0 lies on or within the ellipse is that

d1 + d2 <= dd the distance between p1 an p2 (2*a) which is given by

dd = sqrt((x1 - x2)^2 + (y1 - y2)^2)

This condition can be used directly, but the square roots can be eliminated.

Point.isBetween3 is essentially the same as Point.isBetween2. The code has been modified to eliminate the Math.sqrt calls.

Condition sqrt(a) + sqrt(b) <= sqrt(c), where a >= 0, b >= 0, c >= 0,
is equivalent to condition   0 <= sqr(c) - 2*c*(a+b) + sqr(a-b), where sqr(x) = x^2.

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Proof:
squaring:  a + 2*sqrt(a*b) + b <= c
           2*sqrt(a*b) <= c - (a+b)
squaring:  4*a*b <= sqr(c - (a+b))
           4*a*b <= sqr(c) - 2*c*(a+b) + sqr(a+b)
           4*a*b <= sqr(c) - 2*c*(a+b)) + sqr(a) + 2*a*b + sqr(b)
           0 <= sqr(c) - 2*c*(a+b) + sqr(a) - 2*a*b + sqr(b)
           0 <= sqr(c) - 2*c*(a+b) + sqr(a-b)
QED.
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Let r1 = d1^2, r2 = d2^2 and rr = dd^2.

Substituting r1, r2, rr for a, b, c in the above result we get

0 <= rr^2 - 2*rr*(r1 + r2) + (r1 - r2)^2

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